Math 2040A Tutorial 4th Week

5.2 Diagonalization

Direct Sums.

V — finite dim. vector space.

T — a linear operator on V

Def Let $W_1 \cdots W_k$ be subspaces of V. The sum of these spaces is defined to be the set

$$\{ v_1 + v_2 + \cdots + v_k : v_i \in W_i \quad 1 \le i \le k \}$$

which we denote by $W_1 + \cdots + W_k$ or $\sum W_i$

Ex → Show the $\sum W_i$ is a subspace.

Def Let $W_1 \cdots W_k$ be subspaces of a vector space V.

We call V the direct sum of $W_1 \cdots W_k$ and write $V = W_1 \oplus \cdots \oplus W_k$ if $V = \sum_{i=1}^{k} W_i$ and

$$W_j \cap \sum_{i \ne j} W_i = \{0\} \text{ for each } j \ (1 \le j \le k)$$

Equivalent definitions of a direct sum

Thm 5.10 Let $W_1 \cdots W_k$ be subspaces of V. TFAE:

(a) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

b) $V = \sum_{i=1}^{k} W_i$ and for any vectors $v_1 \cdots v_k$ s.t. $v_i \in W_i$ $(1 \le i \le k)$, if $v_1 + \cdots + v_k = 0$ then $v_i = 0$ for all i.

c) Each vector $v \in V$ can be uniquely written as $v = v_1 + \cdots + v_k$ where $v_i \in W_i$

d) If γ_i is an ordered basis for W_i $(1 \le i \le k)$, then $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for V.

e) For each $i = 1, 2 \cdots, k$, there exists an ordered basis γ_i for W_i s.t. $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for V

(Friedberg ~~P287~~ P276)

Proof (Sketch): $\underline{a) \Rightarrow b)}$

- $V = \sum W_i$ is clear
- $\sum v_i = 0 \Rightarrow v_j = -\sum_{i \ne j} v_i \in W_j \cap \sum_{i \ne j} W_i = \{0\}$

$\underline{b) \Rightarrow c)}$

- $V = \sum W_i \Rightarrow \exists\, v_i \in W_i$ s.t. $v = \sum v_i$
- $\sum v_i = \sum v_i'$ where $v_i, v_i' \in W_i$. $\Rightarrow$

$\sum (v_i - v_i') = 0$ $\quad v_i - v_i' \in W_i \Rightarrow v_i = v_i'$ for all i.

c) $\Rightarrow$ d)

- $V = \sum W_i \Rightarrow \cup \gamma_i$ generates V.
- ($\cup \gamma_i$ is lin. ind.)

$v_{ij} \in \gamma_i$ ($j = 1 \cdots m_i$. $i = 1 \cdots k$)

$\sum a_{ij} v_{ij} = 0$ $\quad w_i := \sum a_{ij} v_{ij} \in W_i \Rightarrow w_i = 0$ for all i

Each γ_i is lin. ind. $\Rightarrow a_{ij} = 0$ for all i, j

d) $\Rightarrow$ e) clear

e) $\Rightarrow$ a)

(Why? cf. P34 Ex 14)

- $V = \text{span}(\cup \gamma_i) = \text{span}\, \gamma_1 + \cdots + \text{span}\, \gamma_k = \sum W_i$

- $v \in W_j \cap \sum_{i \neq j} W_i \Rightarrow v$ can be expressed as a lin. combination of $\cup \gamma_i$ in more than one way.

Contradiction! $\Rightarrow W_j \cap \sum_{i \neq j} W_i = 0$

(Details are presented in class)

<u>Theorem 5.11</u> A lin. op. T on V is diagonalizable iff V is the direct sum of the eigenspaces of V.

Proof: $\lambda_1 \cdots \lambda_k$ – distinct eigenvalues of T.

"$\Rightarrow$" γ_i basis for the eigenspace E_{λ_i}

$\Rightarrow \gamma_1 \cup \cdots \cup \gamma_k$ is a basis for V.

(cf. Thm 5.9)

$\Rightarrow V = \bigoplus E_{\lambda_i}$

"$\Leftarrow$" $V = \bigoplus E_{\lambda_i}$

γ_i ordered basis for E_{λ_i}. $\Rightarrow$

$\cup \gamma_i$ is a basis for V, which consists of eigenvectors of T $\Rightarrow$

T is diagonalizable.

5.1 Eigenvalues

3. eigenvalues — eigenvectors — diagonalize

a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = \mathbb{R}$

Solution: • $\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0$

• $\lambda = 4$: $(A - 4I)x = 0$ infinite solutions (but of 1 dim)

pick one $(2, 3)$.

• $\lambda = 1$: eigenvector $(1, -1)$

• $\beta = \{(2,3), (1,-1)\}$

• $Q = [I]^{\alpha}_{\beta} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ α std basis for $\mathbb{R}^2$

• $D = Q^{-1} A Q = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ $F = \mathbb{R}$

Solution: • $(\lambda - 3)(\lambda - 2)(\lambda - 1) = 0$

• $\lambda = 1$ — $(1, 1, -1)$

• $\lambda = 2$ — $(1, -1, 0)$

• $\lambda = 3$ — $(1, 0, -1)$

- $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

- $D = Q^{-1} A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

12. a) Prove that similar matrices have the same char. poly.
b) Show that the def'n of the char. poly. of a lin. op. on a fin. dim. v.s. V is ind. of the choice of basis for V.

Proof: a) $A = P^{-1} B P$.

$\det(A - \lambda I) = \det(P^{-1}(A - \lambda I) P) = \cdots = \det(B - \lambda I)$

b) choose a different basis for V $=$

conjugate $[T]_\beta^\alpha$ by some $P \in GL(n, F)$.

then use a).

(Details are presented in class)